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Using Muonic Hydrogen in Optical Spectroscopy Experiment to Detect Extra Dimensions

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Considering that gravitational force might deviate from Newton's inverse-square law (ISL) and become much stronger in small scale, we propose a kind of optical spectroscopy experiment to detect this possible deviation and take electronic, muonic and tauonic hydrogen atoms as examples. This experiment might be used to indirectly detect the deviation of ISL down to nanometer scale and to explore the possibility of three extra dimensions in ADD's model, while current direct gravity tests cannot break through micron scale and go beyond two extra dimensions scenario.

KEY WORDS: extra dimensions; ISL; Muonic hydrogen.

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1. INTRODUCTION

In the past few years, many physicists have dedicated themselves to detecting possible deviation of Newton's inverse-square law (ISL) at sub-millimeter scale. These activities are largely due to a work done by N. Arkani-Hamed, S. Dimopoulos and G. Dvali (ADD) in the late 1990s (Arkani-Hamed *et al.*, 1998, 1999; Antoniadis *et al.*, 1998). In ADD's model, the long lasting hierarchy problem, that is, the huge energy gap between Planck scale $M_{\text{pl}} \sim 10^{19}$ Gev and electroweak scale $m_{\text{EW}} \sim 10^3$ Gev, was tentatively solved by assuming that the fundamental energy scale in nature is the $(4+n)$-dimensional Planck scale $M_{\text{pl}(4+n)}$, where n is the number of extra dimensions, while the four-dimensional Planck scale $\sim 10^{19}$ Gev is just an induced one. They further assumed that $M_{\text{pl}(4+n)}$ is around the scale of m_{EW}, and the weakness of four dimensional gravitational force is explained as that the originally strong high dimensional one leaks into extra dimensions. Under the assumption of n equal-radii extra dimensions, they

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deduced that the gravitational force would change from inverse-square law to inverse-$(n+2)$ law if it is measured at a distance far smaller than the radii of extra dimensions R, that is,

$$F \propto \frac{1}{r^2} \to \frac{1}{r^{n+2}}, \quad \text{for} \quad r \ll R. \tag{1}$$

(When $r \gg R$, the ordinary inverse-square law recovers.) In this model, the radii of extra dimensions R is determined by $M_{\mathrm{pl}(4+n)}$ as

$$R = \frac{1}{2\pi} M_{\mathrm{pl}(4+n)}^{-(1+\frac{2}{n})} G_4^{-\frac{1}{n}}, \qquad (c = 1, \quad \hbar = 1) \tag{2}$$

where G_4 is four-dimensional Newton's constant. For $M_{\mathrm{pl}(4+n)} = 1$ Tev,

$$R \approx \frac{1}{\pi} 10^{-17+\frac{32}{n}} \text{ cm}. \tag{3}$$

If $n = 1$, the radius of the extra dimension would be $R \sim 10^{12}$ m. Clearly, this case is ruled out by planetary motion observations. For $n = 2$, $R \sim 10^{-1}$ mm, but gravity measurement had not been done at sub-millimeter scale by the time of the presentation of ADD's model. After a few years' effort (Long *et al.*, 2003; Long and Price, 2003 and an extensive review in Hoyle *et al.*, 2004), under the assumption of two equal-radii extra dimensions, current experiment results require that $M_{\mathrm{pl}(4+n)}$ should not be smaller than about 1 Tev, and the corresponding R (notice Eq. (2)) should not be larger than about 100 μm. (For a recent result, see Kapner *et al.*, 2006; Of course, the probability of one large extra dimension with several small extra dimensions can not be boldly excluded.) Then what about three or more equal-radii extra dimensions? From Eq. (3), one can easily notice that R decreases with the increase of n. If we go one step further, that is, for three equal-radii extra dimensions, we get $R \sim 10^{-7}$ cm. However, direct measurement of gravitational force (e.g. torsion pendulum experiments) at nanometer scale is far beyond the ability of current experiments since the background noises from such as electrostatic, magnetic and Casimir forces would completely swamp gravity effect.

Even if there is no motivation from ADD's proposal, the detection of the possible deviation of ISL in small scale is still meaningful, since there is no special reason to assume that ISL should hold ranging from as large as solar system scale and as small as nucleus scale. Moreover, we should point that except the possible effect of extra dimensions, several other factors could also cause the deviation of ISL in small scale (Adelberger *et al.*, 2003).

At the same time, extensive researches aimed to explore extra dimensions both in theories and experiments were done in the field of accelerators (Macesanu *et al.*, 2002). Constraints on the parameters of extra dimensions, such as $M_{\mathrm{pl}(4+n)}$, have also been given (Luo and Liu, 2005; Mele, 2004).

Considering current direct gravity measurement cannot be performed down to smaller than tens of micron scale, we want to ask if we could find other experimental methods to detect the deviation of ISL at smaller than micron scale and even at nanometer scale? Also, except current known experiments, such as the ones with accelerators and torsion pendulums, if there could be other ways in completely different fields to explore extra dimensions?

In this paper, we try to propose an experimental test in optical spectroscopy to detect the possible deviation of ISL, and we use electronic, muonic and tauonic hydrogen atoms as examples. This proposal is based on a simple idea: if gravity deviates from ISL and becomes much stronger in small scale, then in the field of optical spectroscopy, perhaps the originally safely neglected gravity may need to be considered, since it might be large enough to affect the optical spectrum of the atoms to the extent that we could observe this effect in the lab. This kind of experiments might help to set constraints of $M_{\mathrm{pl}(4+n)}$ or the radii of extra dimensions, and provide a way to further explore the possibility of $n = 3$ in ADD's model.

2. A POSSIBLE OPTICAL SPECTROSCOPY EXPERIMENT TO DETECT THE DEVIATION OF ISL

First, take the ordinary hydrogen atom as an example.

Even if gravity would become stronger in small scale, it is still very weak compared to electromagnetic force. So it is convenient to treat the gravitational potential as a perturbation to calculate the correction of atom's energy levels. Since 1 Gev corresponds to 2.4×10^{23} Hz ($\Delta E = h\Delta\nu$), and the frequencies of the spectrum correspond to the transitions between different energy levels, then comparing the data measured in experiment and the values calculated from Standard Model may provide us some information about the deviation of ISL.

Notice that the correction for the ground state ($1s$) is much larger than for the excited states (since Bohr's radius of ground state is much smaller than the ones of excited states), then as to the orders of magnitude, calculating the correction of the ground state energy is enough if we just want to analyze the possible changes of the spectrum generated from the transitions from some excited states to the ground one.

For definition and simplicity, in the following calculation, we employ ADD's model and assume there exists n equal-radii torus-shaped extra dimensions.

The leading correction comes from the Schrodinger term, that is, for the ground state, we can use the simple perturbation theory of quantum mechanics and obtain the first-order correction of energy for hydrogen atom as

$$\Delta E = (\Psi_{100}, \hat{V}(r)\Psi_{100}), \tag{4}$$

where the wave function of ground state is $\Psi_{100} = \pi^{-\frac{1}{2}} a^{-\frac{3}{2}} e^{-\frac{r}{a}}$, a is Bohr's radius of hydrogen atom $a = \frac{\hbar^2}{me^2}$, m is the reduced mass $m = \frac{m_l m_p}{m_l + m_p}$, m_p is the mass of proton and m_l is the mass of electron—the lightest lepton in nature. Here the gravitational potential is expressed as (Hoyle *et al.*, 2004; Kehagias and Sfetsos, 2000)

$$\hat{V}(r) = \begin{cases} -\frac{G_{(4+n)} m_p m_l}{r^{n+1}}, & r \ll R \\ -\frac{G_4 m_p m_l}{r}\left(1 + \alpha e^{-\frac{r}{\lambda}}\right), & r \sim R. \\ -\frac{G_4 m_p m_l}{r}, & r \gg R \end{cases} \tag{5}$$

The $(4+n)$ Newton's constant is expressed as

$$G_{(4+n)} = \frac{2V_n G_4}{S_n}, \tag{6}$$

where $V_n = (2\pi R)^n$ is the volume of n torus and $S_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})}$ is its surface area ($\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$). Notice that for $r \sim R$, we use the familiar Yukawa potential. For the shape of extra dimensions we mentioned above, we know that $\alpha = \frac{8n}{3}$ and $\lambda = R$. If there is no correction of the gravitational potential from extra dimensions, that is, for $n = 0$, we get

$$\Delta E = -\frac{G_4 m_p m_l}{a}, \tag{7}$$

which is just the correction from the consideration of the simple semi-classical Bohr's hydrogen model.

For $n = 2$, we get (notice Eq. (2))

$$\Delta E \approx \frac{2m_p m_l M_{\text{pl}(4+n)}^{-4}}{\pi a^3} \left[\left(\ln\left(\frac{2r_{\text{pro}}}{a}\right) + C\right)\right], \tag{8}$$

where C is the Euler-Mascheroni constant $C = 0.577215\ldots$, and r_{pro} is of a value about proton's charge radius. For $n = 3$, we get

$$\Delta E \approx -\frac{4m_p m_l M_{\text{pl}(4+n)}^{-5}}{\pi^2 a^3 r_{\text{pro}}}. \tag{9}$$

For $n > 3$, the corresponding frequency corrections of spectrum are so small that they are far beyond observation. Since with the increase of n, R becomes smaller, so the distance that the deviated gravitational potential can act is shorter. Consequently, the corrections decrease with the increase of n.

The parameter $M_{\text{pl}(4+n)}$ comes from the effect of extra dimensions, while r_{pro} is from the cutoff in the integration of $\int_{r_{\text{pro}}}^{r \ll R} \frac{e^{-\frac{2r}{a}}}{r^{n+1}} r^2 dr$. Otherwise, for $n \geq 2$, this integration is divergent when $r \to 0$. Notice that the proton is not a point like particle and the probability that the electron appears in the interior of proton is extremely small, so this cutoff is reasonable.

For hydrogen atom (e-p), if $M_{\text{pl}(4+n)} \sim 1\,\text{Tev}$, we get $\Delta\nu \sim 10^{-8}\,\text{Hz}$ with $n = 2$, and $\Delta\nu \sim 10^{-13}\,\text{Hz}$ with $n = 3$. Although these corrections are much larger than the one calculated from the exact ISL ($\Delta\nu \sim 10^{-24}\,\text{Hz}$), it is still too small to be observed in current experiments.

What's worse, the calculation of the spectrum from the Standard Model is not accurate "enough." For example, the experimental value of hyperfine structure of hydrogen atom (21 cm line) is 1420.4057517667(9) MHz (Hellwig *et al.*, 1970), while the most complete theoretical result (considering reduced mass, QED, and hadronic effects) is 1420.4051(8) MHz (Sapirstein, 1995). So, even this small effect lies within the accuracy of the experiment, there is every reason that it would be swamped by the small "inconsistency" of the data given by the Standard Model and the observation. In other word, we must find some "large" correction that cannot be simply explained out by the inconsistency of the accuracy of the experimental data and Standard Model theoretical values.

Fortunately, nature gives us three generations of leptons. We can treat muon and tau as "heavy" electrons and do the similar calculation using muonic hydrogen (μ-p) and tauonic hydrogen (τ-p). However, we should point out that for μ-p and τ-p, since the corresponding Bohr's radii are far smaller than e-p, the influence from the proton is much more important. (In fact, μ-p has been used to research nucleus for several decades Hughes and Wu, 1977; Indelicato, 2004) It is even likely that the effects of gravity would be submerged by our incomplete knowledge of proton's structure (about its charge and gluon distribution etc.) and other uncertainties. Moreover, as to the spectroscopy experiment itself, the measurement of the transition energies we mentioned above may also not accurate enough. Consequently, perhaps it is better to study the well researched spectrum, such as the hyperfine structure, or to use other atoms' spectrum to extract the possible information about extra dimensions. In this sense, the calculation below may merely a rough description about the orders of magnitude of the corrections.

Anyway, if we use the simple perturbation theory, the main correction is still from the Schrodinger term. From Eqs. (8) and (9), we can easily find that ΔE, or say, $\Delta\nu$ is relevant to the mass of lepton as $\Delta\nu \propto m_l^4$, since $a = \frac{\hbar^2}{me^2}$ and $m \approx m_l$. Because the masses of muon and tau are representatively about 200 and 3000 times of the mass of electron, the corrections of the frequencies for μ-p and τ-p are much larger.

For instance, if we choose $r_{\text{pro}} = 10^{-13}\,\text{cm}$ and $M_{\text{pl}(4+n)} = 1\,\text{Tev}$, for μ-p, we have $\Delta\nu \sim 10^1\,\text{Hz}$ with $n = 2$ and $\Delta\nu \sim 10^{-4}\,\text{Hz}$ with $n = 3$; for τ-p, $\Delta\nu \sim 10^4\,\text{Hz}$ with $n = 2$ and $\Delta\nu \sim 10^0\,\text{Hz}$ with $n = 3$.

The adjustable parameter in the formula ΔE (also $\Delta\nu$) is $M_{\text{pl}(4+n)}$. Consequently, in the field of optical spectroscopy, comparing the theoretical calculations and experimental data might help to set constraint of $M_{\text{pl}(4+n)}$.

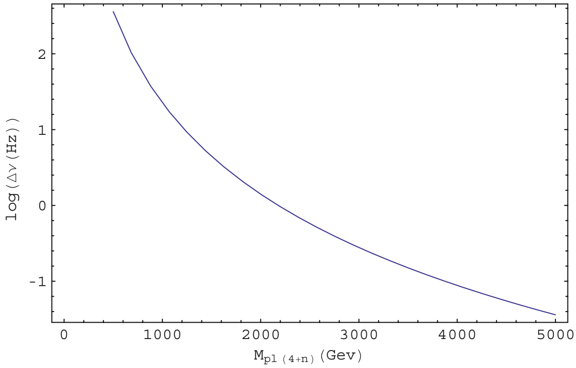

Fig. 1. μ-p ($n = 2$, $r_{\text{pro}} = 10^{-13}$ cm). From Eq. (2), when $M_{\text{pl}(4+n)}$ ranges from 500 Gev to 5000 Gev, R ranges from 1500 μm to 15 μm. (Note: the logarithm is 10-based).

It is easy to see the sensitivity of frequency dependent on $M_{\text{pl}(4+n)}$ and the corresponding R (notice Eq. (2)) in the following figures (Fig. 1). (Considering μ-p is better researched, we only draw figures for μ-p.) From Fig. 2, when $M_{\text{pl}(4+n)}$ varies one order of magnitude, $\Delta\nu$ varies five orders of magnitude.

At last, we would like to point that just like $n = 1$ case can be excluded by the observations of celestial body motions, this possibility may also be ruled out by optical spectroscopy experiments. Since for e-p, $\Delta\nu \sim 10^{-1}$ Hz; for μ-p, $\Delta\nu \sim 10^5$ Hz; and for τ-p, $\Delta\nu \sim 10^8$ Hz. These corrections may have already lain in the accuracy of experiments but waiting for the calculation from Standard Model.

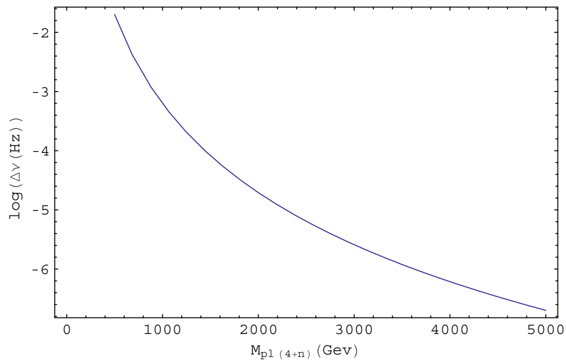

Fig. 2. μ-p ($n = 3$, $r_{\text{pro}} = 10^{-13}$ cm). From Eq. (2), when $M_{\text{pl}(4+n)}$ ranges from 500 Gev to 5000 Gev, R ranges from 5.3 nm to 0.1 nm. (Note: the logarithm is 10-based).

3. CONCLUSIONS AND DISCUSSIONS

We propose an experimental test in optical spectroscopy to indirectly detect the deviation of ISL. One of the most attractive advantages is that this indirect experiment might be used to detect the deviation of ISL down to nanometer scale, while current direct gravitation experiments hardly break through micron scale. Besides, within ADD's model and under the assumption of n equal-radii torus-shaped extra dimensions, using simple perturbation theory in quantum mechanics, we give the corrections of transition frequencies for electronic, muonic and tauonic hydrogen atoms due to the deviation of ISL. This experiment might help to set constraints on $M_{\mathrm{pl}(4+n)}$ and the corresponding radii of extra dimensions R since the corrections of frequencies are sensitive to $M_{\mathrm{pl}(4+n)}$. Moreover, this possible optical spectroscopy experiment may be used to explore the possibility of three equal radii extra dimensions ($n = 3$), while current direct gravity experiments go no further of $n = 2$.

Finally, we have to say that although the accuracy needed for this kind of experiment is a real challenge for experimentalists, perhaps its feasibility is mainly limited by theoretical calculation. Just like the story of the hyperfine structure of hydrogen atom we mentioned above, the experimentalists may have to wait for their fellow theorists providing enough accurate theoretical values to compare with their experimental data. Moreover, the atoms we suggest may be not proper to be used in this kind of experiment. Since many problems, such as the lack of knowledge of proton structure, would seriously impact the accuracy of the calculation and swamp the small effect we suggest, using other kinds of atoms and study different spectrums may be more realizable. In this sense, our proposal just aims to inspire people that perhaps there are methods beside the well-known ways (e.g. accelerators and torsion pendulums) to explore the mysterious extra dimensions.

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